ON CONVEX TO PSEUDOCONVEX MAPPINGS

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Abstract. In the works of Darboux and Walsh, see [D, W], it was remarked that a one to one self mapping of $\mathbb{R}^3$ which sends convex sets to convex ones is affine. It can be remarked also that a $\mathcal{C}^2$-diffeomorphism $F : U \to U^{'}$ between two domains in $\mathbb{C}^n$, $n \geqslant 2$, which sends pseudoconvex hypersurfaces to pseudoconvex ones is either holomorphic or antiholomorphic.

In this note we are interested in the self mappings of $\mathbb{C}^n$ which send convex hypersurfaces to pseudoconvex ones. Their characterization is the following: *A $\mathcal{C}^2$ - diffeomorphism $F : U' \to U$ (where $U', U \subset \mathbb{C}^n$ are domains) sends convex hypersurfaces to pseudoconvex ones if and only if the inverse map $\Phi := F^{-1}$ is weakly pluriharmonic,* i.e., *it satisfies some nice second order PDE very close to $\partial\bar\partial\Phi = 0$.* In fact all pluriharmonic Φ-s do satisfy this equation, but there are also other solutions.

1. Formulation

Let U', U be domains in $\mathbb{C}^n, n \geqslant 2$ and let $F : U' \to U$ be a $\mathcal{C}^2$-diffeomorphism. Coordinates in the source we denote by $z' = x' + iy'$, in the target by $z = x + iy$. It will be convenient for us to suppose that U' is a convex neighborhood of zero and that $F(0') = 0$. The, somewhat unusual choice to put primes on the objects in the source (and not in the target) is explained by the fact that in the statements and in the proofs we shall work more with the inverse map Φ then with F.

Theorem 1. *Let $F : U' \to U$ be a $\mathcal{C}^2$-diffeomorphism. Then the following conditions are equivalent:*

i) For every convex hypersurface $M' \subset U'$ the image $M = F(M')$ is a pseudoconvex hypersurface in U.

ii) The inverse map $\Phi := F^{-1} : U \to U'$ satisfies the following second order PDE System

$$\partial\bar\partial\Phi = (d\Phi^{-1}(\Delta\Phi), dz) \wedge \partial\Phi + (dz, d\Phi^{-1}(\Delta\Phi)) \wedge \bar\partial\Phi. \tag{1.1}$$

iii) The equation (1.1) has the following geometric meaning: for every $z \in U$ and every $\zeta \in T_z\mathbb{C}^n$

$$\partial\bar\partial\Phi_z(\zeta, \bar\zeta) \in \boldsymbol{span}\{d\Phi_z(\zeta), d\Phi_z(i\zeta)\}. \tag{1.2}$$

Here we use the following notation: for a vector $v = (v^1, ..., v^n) \in \mathbb{C}^n$ and $dz = (dz_1, ..., dz_n)$ we set $(dz, v) = \bar v^j dz_j$ and $(v, dz) = v^j d\bar z_j$. Throughout this note we shall use the Einstein summation convention.

Remark 1. Pluriharmonic Φ-s clearly satisfy (1.1) (or (1.2)) and let us remark that this geometric characterization of pluriharmonic diffeomorphisms perfectly agrees with an analytic one: *The class $\mathcal{P}$ of pluriharmonic diffeomorphisms $\mathbb{C}^n \to \mathbb{C}^n$ is stable under*

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biholomorphic parametrization of the source and $\mathbb{R}$-linear of the target. Really, these parametrization preserve accordingly pseudoconvexity and convexity of hypersurfaces.

2. The item *(i)* of the Theorem is clearly equivalent to the following one: *For every strictly convex quadric $M'\cap U'\neq\varnothing$ the image $M=F(M'\cap U')$ is a pseudoconvex hypersurface in U.* I.e, it is enough to check this condition only for quadrics.

3. The fact that (1.1) admits other solutions then just pluriharmonic mappings is very easy to see from the form of its linearization at identity

$$\partial\bar\partial\Phi=(\Delta\Phi,dz)\wedge dz. \tag{1.3}$$

Remark that any map of the form $\Phi(z)=(\varphi_1(z_1),...,\varphi_n(z_n))$ satisfies (1.3) provided all φ_j, except for some j_0, are harmonic. And this φ_{j_0} can be then an arbitrary $\mathcal{C}^2$-function.

2. An auxiliary computation

Denote by $\zeta=\xi+i\eta$ a tangent vector at point $z\in\mathbb{C}^n$. Recall that the real Hessian of a real valued function ρ in $\mathbb{C}^n=\mathbb{R}^{2n}$ is

$$H^{\mathbb{R}}_{\rho(z)}(\zeta,\zeta)=\frac{\partial^2\rho(z)}{\partial x_i\partial x_j}\xi_i\xi_j+\frac{\partial^2\rho(z)}{\partial y_i\partial y_j}\eta_i\eta_j+2\frac{\partial^2\rho(z)}{\partial x_i\partial y_j}\xi_i\eta_j. \tag{2.1}$$

A hypersurface $M=\{z\in U:\rho(z)=0\}$, with ρ is $\mathcal{C}^2$-regular, $\rho(0)=0$ and $\nabla\rho|_M\neq 0$, is strictly convex if the defining function ρ can be chosen with positive definite Hessian, *i.e.*, $H^{\mathbb{R}}_{\rho(z)}(\zeta,\zeta)>0$ for all $z\in M$ and all $\zeta\neq 0$. One readily checks the following expression of the real Hessian of ρ in complex coordinates

$$H^{\mathbb{R}}_{\rho(z)}(\zeta,\zeta)=\frac{\partial^2\rho(z)}{\partial z_i\partial z_j}\zeta_i\zeta_j+\frac{\partial^2\rho(z)}{\partial\bar z_i\partial\bar z_j}\bar\zeta_i\bar\zeta_j+2\frac{\partial^2\rho(z)}{\partial z_i\partial\bar z_j}\zeta_i\bar\zeta_j. \tag{2.2}$$

Recall that the Hermitian part $L_{\rho(z)}(\zeta,\bar\zeta)=\frac{\partial^2\rho}{\partial z_i\partial\bar z_j}\zeta_i\bar\zeta_j$ of the Hessian is called the Levi form of ρ (and of M). M is strictly pseudoconvex if its Levi form is positive definite on the complex tangent space $T^c_zM=\{\zeta\in T_z\mathbb{C}^n:\left(\bar\partial\rho(z),\zeta\right)=0\}$ for every $z\in M$. Here $(\cdot,\cdot)$ stands for the standard Hermitian scalar product in $\mathbb{C}^n$.

Let $F:\mathbb{C}^n_{z'}\supset U'\to U\subset\mathbb{C}^n_z$ be a $\mathcal{C}^2$-diffeomorphism. Let further $z'=z'(z)$ be the coordinate representation of the inverse mapping $z'=\Phi(z):=F^{-1}(z)$ and let $M=F(M')\subset U$ be the image of a hypersurface $M'\subset U'$. Then $M=\{z:\rho(z)=0\}$, where $\rho(z):=\rho'(z'(z))$.

Lemma 2.1. *The Levi form of ρ at point z decomposes as*

$$L_{\rho(z)}(\zeta,\bar\zeta)=L^0_{\rho(z)}(\zeta,\bar\zeta)+L^1_{\rho(z)}(\zeta,\bar\zeta), \tag{2.3}$$

where

$$L^0_{\rho(z)}(\zeta,\bar\zeta)=\frac{1}{4}H^{\mathbb{R}}_{\rho'(z')}\left(d\Phi_z(\zeta),d\Phi_z(\zeta)\right)+\frac{1}{4}H^{\mathbb{R}}_{\rho'(z')}\left(d\Phi_z(i\zeta),d\Phi_z(i\zeta)\right) \tag{2.4}$$

and

$$L^1_{\rho(z)}(\zeta,\bar\zeta)=2\left\langle\nabla\rho'(z'),\partial\bar\partial\Phi_z(\zeta,\bar\zeta)\right\rangle=2Re\left(\bar\partial\rho'(z'),\partial\bar\partial\Phi_z(\zeta,\bar\zeta)\right). \tag{2.5}$$

Proof. Here we denote by $d\Phi_z$ is the differential of the inverse map $\Phi:=F^{-1}$ at point z, $\nabla\rho'(z')$ the real gradient of ρ' at z', $\langle\cdot,\cdot\rangle=Re(\cdot,\cdot)$ - the standard Euclidean scalar product in $\mathbb{C}^n$.

Denote by ν the vector with components $\nu_j = \frac{\partial z'_j}{\partial z_\alpha}\zeta_\alpha$ and by μ with $\mu_j = \frac{\partial z'_j}{\partial \bar z_\alpha}\bar\zeta_\alpha$, *i.e.*,

$$\nu = \partial \Phi_z(\zeta) \;\text{ and }\; \mu = \overline{\partial}\Phi_z(\zeta).$$

Remark that

$$\nu + \mu = d\Phi_z(\zeta) \;\text{ and }\; i(\nu-\mu) = d\Phi_z(i\zeta). \tag{2.6}$$

Write

$$L_{\rho(z)}(\zeta,\bar\zeta) = \frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_\beta}\zeta_\alpha\bar\zeta_\beta = \frac{\partial}{\partial z_\alpha}\left(\frac{\partial\rho'}{\partial z'_i}\frac{\partial z'_i}{\partial\bar z_\beta} + \frac{\partial\rho'}{\partial\bar z'_i}\frac{\partial\bar z'_i}{\partial\bar z_\beta}\right)\zeta_\alpha\bar\zeta_\beta =$$
$$= \frac{\partial^2\rho'}{\partial z'_i\partial z'_j}\frac{\partial z'_i}{\partial\bar z_\beta}\frac{\partial z'_j}{\partial z_\alpha}\zeta_\alpha\bar\zeta_\beta + \frac{\partial^2\rho'}{\partial\bar z'_i\partial\bar z'_j}\frac{\partial\bar z'_i}{\partial\bar z_\beta}\frac{\partial\bar z'_j}{\partial z_\alpha}\zeta_\alpha\bar\zeta_\beta + \frac{\partial^2\rho'}{\partial z'_i\partial\bar z'_j}\left(\frac{\partial z'_i}{\partial\bar z_\beta}\frac{\partial\bar z'_j}{\partial z_\alpha} + \frac{\partial z'_i}{\partial z_\alpha}\frac{\partial\bar z'_j}{\partial\bar z_\beta}\right)\zeta_\alpha\bar\zeta_\beta +$$
$$+\left(\frac{\partial\rho'}{\partial z'_i}\frac{\partial^2 z'_i}{\partial z_\alpha\partial\bar z_\beta} + \frac{\partial\rho'}{\partial\bar z'_i}\frac{\partial^2\bar z'_i}{\partial z_\alpha\partial\bar z_\beta}\right)\zeta_\alpha\bar\zeta_\beta = \frac{\partial^2\rho'}{\partial z'_i\partial z'_j}\mu_i\nu_j + \frac{\partial^2\rho'}{\partial\bar z'_i\partial\bar z'_j}\bar\nu_i\bar\mu_j + \frac{\partial^2\rho'}{\partial z'_i\partial\bar z'_j}\left[\mu_i\bar\mu_j + \nu_i\bar\nu_j\right] +$$
$$+\left(\frac{\partial\rho'}{\partial z'_i}\frac{\partial^2 z'_i}{\partial z_\alpha\partial\bar z_\beta} + \frac{\partial\rho'}{\partial\bar z'_i}\frac{\partial^2\bar z'_i}{\partial z_\alpha\partial\bar z_\beta}\right)\zeta_\alpha\bar\zeta_\beta = L^0_{\rho(z)}(\nu,\mu) + L^1_{\rho(z)}(\zeta,\bar\zeta)$$

with

$$L^0_{\rho(z)}(\nu,\mu) = \frac{\partial^2\rho'}{\partial z'_i\partial z'_j}\nu_i\mu_j + \frac{\partial^2\rho'}{\partial\bar z'_i\partial\bar z'_j}\bar\nu_i\bar\mu_j + \frac{\partial^2\rho'}{\partial z'_i\partial\bar z'_j}\left[\mu_i\bar\mu_j + \nu_i\bar\nu_j\right] \tag{2.7}$$

and

$$L^1_{\rho(z)}(\zeta,\bar\zeta) = \left(\frac{\partial\rho'}{\partial z'_i}\frac{\partial^2 z'_i}{\partial z_\alpha\partial\bar z_\beta} + \frac{\partial\rho'}{\partial\bar z'_i}\frac{\partial^2\bar z'_i}{\partial z_\alpha\partial\bar z_\beta}\right)\zeta_\alpha\bar\zeta_\beta. \tag{2.8}$$

We need to get more information about the structure of both terms L^0_ρ and L^1_ρ of the Levi form. Let's prove that the following relation holds

$$L^0_{\rho(z)}(\nu,\mu) = \frac{1}{4}H^{\mathbb{R}}_{\rho'(z')}\left(\nu+\mu,\nu+\mu\right) + \frac{1}{4}H^{\mathbb{R}}_{\rho'(z')}\left(i(\nu-\mu), i(\nu-\mu)\right). \tag{2.9}$$

To see this we make the following change in (2.9):

$$\mu_j = V_j + iW_j,\; \nu_j = V_j - iW_j.$$

or

$$V = \frac{1}{2}(\nu+\mu) = \frac{1}{2}d\Phi_z(\zeta),\; W = \frac{i}{2}(\nu-\mu) = \frac{1}{2}d\Phi_z(i\zeta). \tag{2.10}$$

Then

$$L^0_{\rho(z)}(\nu,\mu) = \frac{\partial^2\rho'}{\partial z'_i\partial z'_j}\left(V_i - iW_i\right)\left(V_j + iW_j\right) + \frac{\partial^2\rho'}{\partial\bar z'_i\partial\bar z'_j}\left(\overline{V_i} + i\overline{W_i}\right)\left(\overline{V_j} - i\overline{W_j}\right) +$$
$$+\frac{\partial^2\rho'}{\partial z'_i\partial\bar z'_j}\left[\left(V_i + iW_i\right)\left(\overline{V_j} - i\overline{W_j}\right) + \left(V_i - iW_i\right)\left(\overline{V_j} + i\overline{W_j}\right)\right] =$$
$$= \frac{\partial^2\rho'}{\partial z'_i\partial z'_j}\left(V_iV_j + W_iW_j\right) + i\frac{\partial^2\rho'}{\partial z'_i\partial z'_j}\left(V_iW_j - W_iV_j\right) + \frac{\partial^2\rho'}{\partial\bar z'_i\partial\bar z'_j}\left(\overline{V_iV_j} + \overline{W_iW_j}\right) +$$
$$+i\frac{\partial^2\rho'}{\partial\bar z'_i\partial\bar z'_j}\left(\overline{W_iV_j} - \overline{V_iW_j}\right) + 2\frac{\partial^2\rho'}{\partial z'_i\partial\bar z'_j}\left(V_i\overline{V_j} + W_i\overline{W_j}\right) = H^{\mathbb{R}}_{\rho'(z')}(V,V) + H^{\mathbb{R}}_{\rho'(z')}(W,W).$$

We used the obvious relations $\frac{\partial^2\rho'}{\partial z'_i\partial z'_j}\left(V_iW_j - W_iV_j\right) = 0 = \frac{\partial^2\rho'}{\partial\bar z'_i\partial\bar z'_j}\left(\overline{W_iV_j} - \overline{V_iW_j}\right)$ and the complex expression of the real Hessian (2.2). Therefore

$$L^0_{\rho(z)}(\nu,\mu) = H^{\mathbb{R}}_{\rho'(z')}(V,V) + H^{\mathbb{R}}_{\rho'(z')}(W,W). \tag{2.11}$$

From (2.10) and (2.11) we get the formula (2.4) of the Lemma.

Remark 2. If the real Hessian of ρ' at z' is positive (resp. non-negative) definite then the component $L^0_{\rho(z)}(\nu,\mu)$ of the Levi form of ρ at $z = F(z')$ is also positive (resp. non-negative) definite for any $\mathcal{C}^2$-germ of a diffeomorphism F.

Now we turn to L^1_ρ. Note that in complex notations $\nabla\rho = \bar{\partial}\rho$ as well as that standard Euclidean scalar product $\langle\cdot,\cdot\rangle$ in $\mathbb{C}^n$ is equal to the real part of the Hermitian one $(\cdot,\cdot)$. Therefore from (2.8) we get

$$L^1_{\rho(z)}(\zeta,\bar{\zeta}) = \left(\frac{\partial\rho'}{\partial z'_i}\frac{\partial^2 z'_i}{\partial z_\alpha\partial\bar{z}_\beta} + \frac{\partial\rho'}{\partial\bar{z}'_i}\frac{\partial^2\bar{z}'_i}{\partial z_\alpha\partial\bar{z}_\beta}\right)\zeta_\alpha\bar{\zeta}_\beta =$$

$$= \overline{\left(\bar{\partial}\rho', \frac{\partial^2 z'}{\partial\bar{z}_\alpha\partial z_\beta}\bar{\zeta}_\alpha\zeta_\beta\right)} + \left(\bar{\partial}\rho', \frac{\partial^2 z'}{\partial\bar{z}_\alpha\partial z_\beta}\bar{\zeta}_\alpha\zeta_\beta\right) = \overline{\left(\bar{\partial}\rho', \partial\bar{\partial}\Phi_z(\zeta,\bar{\zeta})\right)}$$

$$+ \left(\bar{\partial}\rho', \partial\bar{\partial}\Phi_z(\zeta,\bar{\zeta})\right) = 2Re\left(\bar{\partial}\rho', \partial\bar{\partial}\Phi_z(\zeta,\bar{\zeta})\right) = 2\left\langle\nabla\rho', \partial\bar{\partial}\Phi_z(\zeta,\bar{\zeta})\right\rangle,$$

which proves (2.5).

$\square$

3. Proof of the Theorem

(i) $\Longleftrightarrow$ *(iii)*

We start with the proof of the geometric characterization of convex to pseudoconvex mappings given in *(iii)* of the Theorem. By a complex (real) line in $\mathbb{C}^n$ we mean an 1-dimensional complex (real) subspace of $\mathbb{C}^n$. The same for complex (real) plain. Take a complex line $l = \mathtt{span}\{\zeta\}$ in $T_z\mathbb{C}^n$ and let $\Pi' \subset T_{z'}\mathbb{C}^n$ be the real plain - image of l under $d\Phi_z$, *i.e.*, $\Pi' = \mathtt{span}\{d\Phi_z(\zeta), d\Phi_z(i\zeta)\}$. Let $l' := \partial\bar{\partial}\Phi_z(l)$ denotes the ***real*** (!) line - image of l under the mapping

$$\partial\bar{\partial}\Phi_z : \mathbb{C}^n_z \to \mathbb{C}^n_{z'},$$

defined as

$$\zeta \mapsto \partial\bar{\partial}\Phi(\zeta,\bar{\zeta}) := \frac{\partial^2\Phi(z)}{\partial z_\alpha\bar{\partial}z_\beta}\zeta_\alpha\bar{\zeta}_\beta.$$

We consider l' as a real line in $T_{z'}\mathbb{C}^n$.

Lemma 3.1. *Suppose that given a diffeomorphism $F : U' \to U$. Then F sends convex quadrics to pseudoconvex hypersurfaces if and only if for every $z \in U$ and for all $l' := \partial\bar{\partial}\Phi_z(l)$ and $\Pi' = d\Phi_z(l)$ as above one has $l' \subset \Pi'$.*

Proof. Let us prove the "only if " assertion first. We may suppose that $z' = 0'$. Take any strictly convex $M' = \{\rho'(z') = 0\}$ defined by a $\mathcal{C}^2$-function ρ' with positive defined Hessian such that $T_{0'}M' \supset \Pi'$. By M denote the image $F(M')$.

Consider the following family of hypersurfaces in U': $M'_t = \{z : \rho'_t(z') := \rho'(z') + t\langle\nabla\rho'(0'), z'\rangle = 0\}$, $t \in \mathbb{R}$, $t \neq -1$. All M'_t are strictly convex (they have the same quadratic part as M'), all path through zero and $M'_0 = M'$. In addition all M'_t are smooth at zero with $T_0M'_t = T_{0'}M'$ for all $t \neq -1$, because $\nabla\rho'_t(0') = (1+t)\nabla\rho'(0')$. Moreover, if we take some $\zeta \in T^c_0M$ then ζ will stay to be complex tangent to all $M_t := F(M'_t)$ at zero because $T_0M_t = dF_{0'}(T_{0'}M'_t)$ is the same for all t. From Lemma 2.1 we see that

$$L_{\rho_t(0)}(\zeta,\bar\zeta)=L^0_{\rho(0)}(\zeta,\bar\zeta)+2(1+t)\left\langle\nabla\rho'(0'),\partial\bar\partial\Phi(0)(\zeta,\bar\zeta)\right\rangle, \tag{3.1}$$

because $L^0_{\rho_t}(0)(\zeta,\bar\zeta)=L^0_{\rho}(0)(\zeta,\bar\zeta)$ for all t due to the fact that coefficients of L^0_ρ depend only on the second derivatives of ρ' at $0'$ and on $d\Phi_0$.

Suppose $\left\langle\nabla\rho'(0'),\partial\bar\partial\Phi(0)(\zeta,\bar\zeta)\right\rangle\neq 0$. Then taking an appropriate t_0 we can make $L_{\rho_t(0)}(\zeta,\bar\zeta)=0$ because $L^0_{\rho(0)}(\zeta,\bar\zeta)$ do not depend on t. Remark that $t_0\neq -1$ because $L^0_{\rho(0)}(\zeta,\bar\zeta)>0$. Now we can deform M'_t letting t run over a neighborhood of t_0. M'_t stays strictly convex while the Levi form of M_t changes its sign on the vector ζ. Contradiction with assumed property of F. Therefore

$$\left\langle\nabla\rho'(0'),\partial\bar\partial\Phi(0)(\zeta,\bar\zeta)\right\rangle=0, \tag{3.2}$$

for every strictly convex $M'=\{z':\rho'(z')=0\}$ such that $T_{0'}M'\supset\Pi'$. For any vector $v\in T_{0'}\mathbb{C}^n$ orthogonal to Π' we can take a strictly convex hypersurface $M'=\{z':\rho'(z')=0\}$ such that $\nabla\rho'(0')=v$. Therefore $\partial\bar\partial\Phi(0)(\zeta,\bar\zeta)$ is orthogonal to every such v. So $l'=\partial\bar\partial\Phi(0)(l)\subset\Pi'$ and the "only if " assertion of the lemma is proved.

To prove the opposite direction take a convex quadric $M'=\{z':\rho'(z')=0\}$ and set $M=F(M')$. Let $\zeta\in T^c_zM$. Use again Lemma 2.1. The term $L^0_{\rho(z)}(\zeta,\bar\zeta)$ is clearly positive. The term $L^1_{\rho(z)}(\zeta,\bar\zeta)$ is zero because $\partial\bar\partial\Phi_z(\zeta,\bar\zeta)\in d\Phi_z(\langle\zeta\rangle)\subset T_{z'}M'$.

□

Let us reformulate the result obtained as follows (and remark that the equivalence of *(i)* and *(iii)* in Theorem is proved):

Corollary 3.1. *If F sends convex quadrics to pseudoconvex hypersurfaces if and only if for every $z\in U$ and every vector $\zeta\in T_z\mathbb{C}^n$ the following holds:*

$$\partial\bar\partial\Phi_z(\zeta,\bar\zeta)\in \textbf{\textit{span}}\{d\Phi_z(\zeta),d\Phi_z(i\zeta)\}. \tag{3.3}$$

For the convenience of future references let us formulate the abovementioned statement about holomorphic mappings:

Corollary 3.2. *A $\mathcal{C}^2$-difeomorphism $F:U'\to U$ sends pseudoconvex quadrics to pseudoconvex hypersurfaces if and only if F is either holomorphic or antiholomorphic.*

Proof. This is well known but still let us give a proof. Suppose, for example, that Φ is antiholomorphic, then $\nu=0$ as defined in (2.6). Therefore (2.5) tells us that $L^1_{\rho(z)}(\zeta',\bar\zeta')\equiv 0$ in the representation (2.3). Now (2.7) shows that and gave us

$$L_{\rho(z)}(\zeta,\bar\zeta)=L_{\rho'(z')}\left(\bar\partial\Phi_z(\zeta),\overline{\bar\partial\Phi_z(\zeta)}\right)$$

for every complex tangent ζ. Conclusion follows.

Suppose that, vice versa, F sends pseudoconvex quadrics to pseudoconvex hypersurfaces. (3.3) shows that $\partial\bar\partial\Phi_z(\zeta,\bar\zeta)$ belongs to the plain $\mathtt{span}\{d\Phi_z(\zeta),d\Phi_z(i\zeta)\}$ for all $\zeta\in\mathbb{C}^n_z$. And therefore for every ζ complex tangent to $M=\{\rho(z)=0\}$ the vector $\partial\bar\partial\Phi_z(\zeta,\bar\zeta)$ is tangent to $M'=\{\rho'(z')=0\}$. Consequently $L^1_{\rho(z)}(\zeta,\bar\zeta)\equiv 0$ for any ρ. Apply (2.9) to the quadric

$$\rho'(z') = \sum_{j=1}^{n}(z_j^2 + \bar{z}_j^2 + \varepsilon|z_j|^2) + L(z) + \overline{L(z)} \tag{3.4}$$

(where $\varepsilon > 0$ and L is a $\mathbb{C}$-linear form) and get

$$L^0_{\rho(z)}\left(\zeta,\bar{\zeta}\right) = \sum_{j=1}^{n}(\nu_j\mu_j + \bar{\nu}_j\bar{\mu}_j + \varepsilon|\nu_j|^2 + \varepsilon|\mu_j|^2) = 2Re\left(\sum_{j=1}^{n}\nu_j\mu_j\right) + \varepsilon(\|\nu\|^2 + \|\mu\|^2). \tag{3.5}$$

Taking different linear forms L in (3.4) we can deploy any $\zeta \in \mathbb{C}^n$ as a complex tangent and therefore, if Φ is neither holomorphic no antiholomorphic, then we see from (2.6) that ν and μ can be taken arbitrary. But for arbitrary taken ν and μ (3.5) cannot be positive. Contradiction.

□

(iii) $\Longleftrightarrow$ *(ii)*

We shall need the following linear algebra lemma. Let V and W be $\mathbb{C}$-linear spaces. We suppose that on V some Hermitian scalar product $(\cdot,\cdot)$ is fixed. Let $B(\zeta,\bar{\eta}) : V \times V \to W$ be a sesquilinear map. Its trace is defined as $TrB = \sum_\alpha B(e_\alpha,\bar{e}_\alpha)$ for an orthonormal frame in $(V,(\cdot,\cdot))$. Let, furthermore $C : V \to W$ be an $\mathbb{R}$-linear isomorphism. Denote by $C^{1,0}$ (resp. $C^{0,1}$) the complex linear (resp. antilinear) part of C.

Lemma 3.2. *The following properties ob the pair (B,C) are equivalent:*

$$B(\zeta,\bar{\zeta}) \in \textbf{\textit{span}}\{C(\zeta),C(i\zeta)\} \quad \text{for all} \quad \zeta \in V. \tag{3.6}$$

$$B(\zeta,\bar{\eta}) = \left(C^{-1}(TrB),\eta\right)C^{1,0}(\zeta) + \left(\zeta,C^{-1}(TrB)\right)C^{0,1}(\eta) \quad \text{for all} \quad \zeta,\eta \in V. \tag{3.7}$$

Proof. Define the induced quadratic map $A : V \to V$ as $A(\zeta,\bar{\zeta}) = C^{-1}\circ B(\zeta,\bar{\zeta})$. Note that A is not sesquilinear in general. Note that the image of every complex line in V under a quadratic map is a real line.

Write (3.6) in the form $A(\zeta,\bar{\zeta}) = k(\zeta)\cdot\zeta$, where k is a complex valued function. One readily sees that $k(\lambda\zeta) = \bar{\lambda}k(\zeta)$. The polarization equality for A

$$A(\zeta+\eta,\bar{\zeta}+\bar{\eta}) + A(\zeta-\eta,\bar{\zeta}-\bar{\eta}) = 2A(\zeta,\bar{\zeta}) + 2A(\eta,\bar{\eta})$$

gives

$$k(\zeta+\eta)(\zeta+\eta) + k(\zeta-\eta)(\zeta-\eta) = 2k(\zeta)\zeta + 2k(\eta)\eta,$$

or, for complex independent vectors

$$k(\zeta+\eta) + k(\zeta-\eta) = 2k(\zeta) \text{ and } k(\zeta+\eta) - k(\zeta-\eta) = 2k(\eta),$$

which implies additivity of k: $k(\zeta+\eta) = k(\zeta) + k(\eta)$ for complex independent ζ,η and, therefore for all. So k is an antilinear form on V and by Ries representation we obtain a vector v such that $k(\zeta) = (v,\zeta)$ for all $\zeta \in V$ and therefore $A(\zeta,\bar{\zeta}) = (v,\zeta)\zeta$ and consequently

$$B(\zeta,\bar{\zeta}) = C\left((v,\zeta)\zeta\right) \quad \text{for all} \quad \zeta \in V. \tag{3.8}$$

Furthermore,

$$B(\zeta,\bar{\eta}) + B(\eta,\bar{\zeta}) = B(\zeta+\eta,\bar{\zeta}+\bar{\eta}) - B(\zeta,\bar{\zeta}) - B(\eta,\bar{\eta}) = C\left((v,\zeta+\eta)(\zeta+\eta)\right) -$$
$$-C\left((v,\zeta)\zeta\right) - C\left((v,\eta)\eta\right) = C\left((v,\eta)\zeta\right) + C\left((v,\zeta)\eta\right).$$

and

$$-iB(\zeta,\bar\eta)+iB(\eta,\bar\zeta)=B(\zeta+i\eta,\bar\zeta-i\bar\eta)-B(\zeta,\bar\zeta)-B(\eta,\bar\eta)=C\left((v,\zeta+i\eta)(\zeta+i\eta)\right)-$$
$$-C\left((v,\zeta)\zeta\right)-C\left((v,\eta)\eta\right)=C\left(-i(v,\eta)\zeta\right)+C\left(i(v,\zeta)\eta\right).$$

Therefore

$$2B(\zeta,\bar\eta)=C\left((v,\eta)\zeta\right)-iC\left(v,\eta)\zeta\right)+C\left((v,\zeta)\eta\right)+iC\left(i((v,\zeta)\eta\right).$$

So we obtain

$$B(\zeta,\bar\eta)=(v,\eta)C^{1,0}(\zeta)+(\zeta,v)C^{0,1}(\eta). \tag{3.9}$$

Set in (3.9) $\zeta=\eta=e_\alpha$. Then

$$TrB=\sum_\alpha B(e_\alpha,\bar e_\alpha)=C^{1,0}\left(\sum_\alpha(v,e_\alpha)\right)+C^{0,1}\left(\sum_\alpha(v,e_\alpha)\right)=C(v).$$

Therefore $v=C^{-1}(TrB)$ and (3.7) is established.

The opposite implication is easy, because (3.7) tells, if η is taken to be equal to ζ, that

$$B(\zeta,\bar\zeta)=aC^{1,0}(\zeta)+\bar aC^{0,1}(\zeta)=a\frac{1}{2}\left(C(\zeta)-iC(i\zeta)\right)+\bar a\frac{1}{2}\left(C(\zeta)+iC(i\zeta)\right)=$$
$$=\mathsf{Re}\,a\cdot C(\zeta)+\mathsf{Im}\,a\cdot C(i\zeta)\in\mathtt{span}\{C(\zeta),C(i\zeta)\}.$$

$\square$

We apply this lemma for $B=\partial\bar\partial\Phi_z:T_z\mathbb{C}^n\to T_{z'}\mathbb{C}^n$, $C=dF_{z'}^{-1}:T_{z'}\mathbb{C}^n\to T_z\mathbb{C}^n$ and, as a result $A=dF_{z'}\circ\partial\bar\partial\Phi_z:T_z\mathbb{C}^n\to T_z\mathbb{C}^n$ for every $z=F(z')$ and get

Corollary 3.3. *A $\mathcal{C}^2$-diffeomorphism F sends convex quadrics to pseudoconvex hypersurfaces if and only if*

$$\partial\bar\partial\Phi_z(\zeta,\bar\eta)=\left(dF_{z'}\left(Tr\partial\bar\partial\Phi_z\right),\eta\right)\partial\Phi(\zeta)+\left(\zeta,dF_{z'}\left(Tr\partial\bar\partial\Phi_z\right)\right)\bar\partial\Phi(\eta) \tag{3.10}$$

for all $z=F(z')$ and all $\zeta,\eta\in T_z\mathbb{C}^n$.

And this is equivalent to (1.1). Theorem is proved.

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